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LAMINAR FREE CONVECTION FROM A NONISOTHERMAL PLATE IMMERSED IN A TEMPERATURE STRATIFIED MEDIUM*

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Abstract—This study deals with natural convection heat transfer from a non-isothermal vertical flat plate immersed in a temperature stratified environment. Extensive numerical calculations based on similarity solutions have been carried out for a wide range of wall and ambient temperature distributions for Prandtl numbers between 0·1 and 20. The limiting cases of the Prandtl number approaching zero and infinity have also been considered. The relative effect of the individual temperature distributions is discussed in detail. The flow reversal and temperature defect behavior found in other investigations dealing with nonuniform ambient temperatures is closely examined. It is also shown that approximations based on the local temperature difference can introduce large errors into the prediction of surface heat transfer rates. Although the magnitude of this error depends on the situation being examined, the procedure for calculating it is clearly discussed.

NOMENCLATURE

f, stream function;

F, stream function for limiting Prandtl number cases, equations (18) and (26);

g, acceleration due to gravity;

G, dimensionless constant, equation (5);

G_w, dimensionless temperature at the wall, equation (5);

G_∞, dimensionless temperature in the environment, equation (6);

Gr_x, local Grashof number, equation (37);

k, thermal conductivity;

L, length of plate;

m, dimensionless constant, equation (5);

n, dimensionless constant, equation (5);

Nu, Nusselt number, hx/k;

Pr, Prandtl number;

q, heat transfer rate;

T, temperature;

u, velocity in x-direction;

v, velocity in y-direction;

x, distance along plate surface;

y, distance perpendicular to plate surface;

α, thermal diffusivity;

β, coefficient of thermal expansion;

η, dimensionless similarity variable, equation (9);

θ, dimensionless temperature, $(T - T_\infty)/(T_w - T_\infty)$;

ν, kinematic viscosity;

ξ, dimensionless variable for limiting Prandtl number cases, equations (17) and (25);

ρ, density;

τ, surface shear stress;

Φ, dimensionless temperature for limiting Prandtl number cases, equations (19) and (27).

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Subscripts

- w, wall value;
- ∞, ambient value;
- r, reference value.

INTRODUCTION

MANY free convection processes occur in environments with temperature stratification. Good examples are closed containers and environmental chambers with heated walls. Also of interest is free convection associated with heat-rejection systems for long-duration deep-ocean power modules where the ocean environment is stratified [1]. To realistically cope with such situations, it is necessary to understand as well as to be able to predict the effect of environmental temperature stratification on surface heat transfer rates. Unfortunately, very little is known in the literature. For laminar free convection along a vertical plate, Cheesewright [2] has obtained similarity solutions dealing with various types of nonuniform ambient temperature distributions by using the technique of Yang [3]; however, only a few cases with numerical results are given. The corresponding nonsimilar problem has been recently analyzed by Eichhorn [4], but only for cases of linear wall and ambient temperature distributions. In both investigations, numerical results show that the temperature can fall below that of the environment and flow reversal can occur in the outer region of the boundary layer. Whether this behavior has significant implication is not clearly discussed by the previous authors. In fact, the relative significance of the wall and ambient temperature distributions with respect to surface heat transfer is still not known.

The objective of this paper is to present the results of an in-depth study of the same problem with particular emphasis on the physical aspects of the phenomena. The analysis is based on a similarity solution, similar to that previously obtained by Cheesewright [2], which is flexible enough to include parameters characterising both nonuniform wall and nonuniform ambient temperature distributions. Extensive numerical calculations have been carried out covering the important ranges of all parameters, including those associated with limiting Prandtl-number analyses.

ANALYSIS

For laminar free convection on a heated vertical plate with variable wall and ambient temperatures, the usual boundary-layer equations may be written as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = g\beta(T - T_\infty) + \nu\frac{\partial^2 u}{\partial y^2} \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \quad (3)$$

where the coordinates are shown in Fig. 1. The

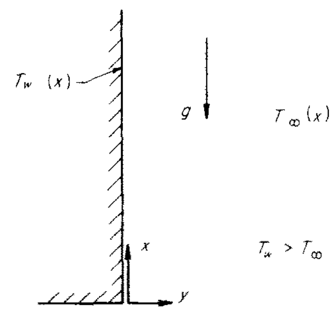

FIG. 1. Physical model.

corresponding boundary conditions are

$$\begin{aligned} y = 0: &\quad u = v = 0, \quad T = T_w(x) \\ y \to \infty: &\quad u \to 0, \quad T \to T_\infty(x). \end{aligned} \quad (4)$$

The usual Boussinesq approximation is retained in the analysis. It is also understood that the ambient temperature variation must be such that the environment is hydrodynamically stable with no motion. In accordance with the technique of Yang [3], it is easily found that for wall and ambient temperature distributions given by

$$G_w = \frac{g\beta L^3}{\nu^2}(T_w - T_r) = G(m+1)\left(\frac{4}{G}\frac{x}{L}\right)^n \quad (5)$$

$$G_\infty = \frac{g\beta L^3}{\nu^2}(T_\infty - T_r) = Gm\left(\frac{4}{G}\frac{x}{L}\right)^n \quad (6)$$

the governing equations reduce to the following two simultaneous ordinary differential equations

$$f''' + (n+3)ff'' - 2(n+1)(f')^2 + \theta = 0 \quad (7)$$

$$\frac{1}{Pr}\theta'' + (n+3)f\theta' - 4f'\theta - 4nmf' = 0 \quad (8)$$

where T_r is any convenient reference temperature, and G, m and n are dimensionless constants. The variable f is a dimensionless stream function, $\theta = (T - T_\infty)/(T_w - T_\infty)$, and the prime refers to a derivative with respect to the similarity variable

$$\eta = \frac{y}{L}\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{n-1}{4}} \quad (9)$$

The boundary conditions given in equation (4) reduce to

$$f(0) = f'(0) = f'(\infty) = \theta(\infty) = 0,$$
$$\theta(0) = 1{\cdot}0. \quad (10)$$

It is of interest to note that the driving force is still the local temperature difference across the boundary layer and that the effect of ambient temperature variation is completely governed by the single term $4mnf'$ in the transformed energy equation. For $m = 0$, the ambient temperature T_∞ in equation (6) is constant and the problem reduces to the one treated by Sparrow and Gregg [5]. Once all of the constant parameters (Pr, m and n) are specified, equations (7) and (8) can be solved numerically to yield velocity and temperature profiles:

$$\frac{uL}{\nu} = G\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{n+1}{2}} f' \quad (11)$$

$$\frac{vL}{\nu} = -\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{n-1}{4}}[(n+3)f + (n-1)\eta f'] \quad (12)$$

$$\frac{T - T_\infty}{T_w - T_\infty} = \theta. \quad (13)$$

In turn, the surface shear stress and heat transfer are given by

$$\frac{\tau L^2}{\rho\nu^2} = G\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{3n+1}{4}} f''(0) \quad (14)$$

$$q\frac{\beta g L^4}{\nu^2 k} = -G\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{5n-1}{4}} \theta'(0). \quad (15)$$

The constants T_r and G specify the temperature and Grashof number levels respectively, and are just scaling factors in accordance with the concept of a boundary layer. The constants m, n and Pr are the true parameters in the problem. Although both m and n determine the manner in which the wall temperature and ambient temperature vary along the plate, the constant n governs the local temperature difference since

$$G_w - G_\infty = g\frac{\beta L^3}{\nu^2}(T_w - T_\infty)$$
$$= G\left(\frac{4}{G}\frac{x}{L}\right)^n. \quad (16)$$

Figure 2 presents a pictorial representation of the wall and ambient temperature distributions as functions of m and n. Several special combinations of m and n are of physical interest. As mentioned, $m = 0$ signifies the case of constant ambient temperature [5]. The case $n = 0$ is identical to the classical Pohlhausen solution. For $m = -1$, the wall temperature is constant while the ambient temperature varies. In addition, equations (14) and (15) show that regardless of value of m, the case of uniform wall shear is given by $n = -\frac{1}{3}$, while the case of constant wall heat flux is given by $n = \frac{1}{5}$. Also, the wall and

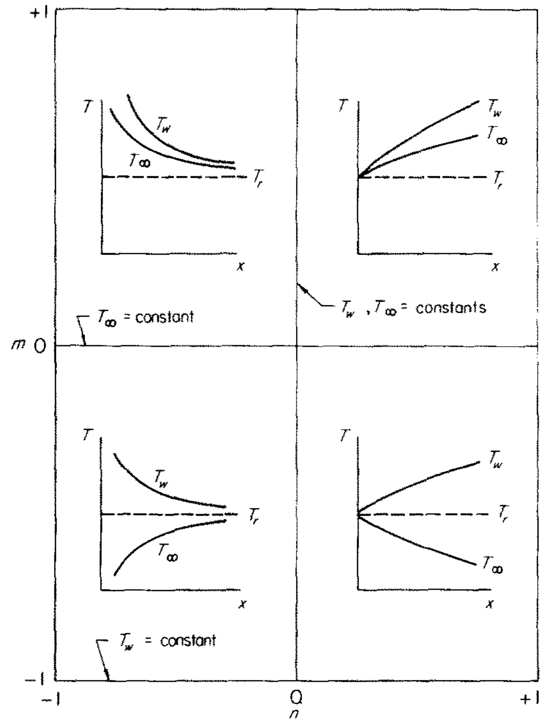

FIG. 2. Pictorial representation of the wall and the ambient temperature distributions as a function of m and n.

ambient temperature distributions are linear for $n = 1$. Furthermore, for $n < 0$, the local temperature differences as well as the ambient and wall temperatures increase without bound as the leading edge is approached from above. Except for qualitative purposes, these latter cases are obviously physically unrealistic. For $m < 0$, $n < 0$ and $m >$, $n > 0$, the environment could be considered as heated from below, while the reverse is true in the other two quadrants. It is clear that this similarity solution is significant since it is capable of describing a great variety of wall and ambient temperature distributions. The major objective of this study is to make sufficient numerical calculations for adequate ranges of m and n values so that the results can be used to determine the relative significance of the temperature distributions on free convection.

In a convective heat transfer study, it is always desirable to determine the influence of the Prandtl number in the limit of almost zero for liquid metals to the limit of very large values for high-viscosity fluids. Analyses are presented to investigate the limiting behavior of free convection in stratified fluids as $Pr \to 0$ and $Pr \to \infty$. Similar analyses for simpler free-convection cases are given by Le Fevre [6] and Kuiken [7]; thus, it is sufficient to present the resulting equations for the limiting cases. Introducing

$$\eta = (n + 3)^{-\frac{1}{4}}(Pr)^{-\frac{1}{4}}\xi \quad (17)$$

$$f(\eta) = (n + 3)^{-\frac{3}{4}}(Pr)^{-\frac{1}{4}}F(\xi) \quad (18)$$

$$\theta(\eta) = \Phi(\xi) \quad (19)$$

for $Pr \to 0$, equations (7) and (8) reduce respectively to

$$FF'' - 2\frac{(n + 1)}{n + 3}(F')^2 + \Phi = 0 \quad (20)$$

$$\Phi'' + F\Phi' - \frac{4n}{n+3}(F')\Phi - \frac{4nm}{n+3}(F') = 0 \quad (21)$$

where the prime refers to a derivative with respect to ξ. The corresponding boundary conditions are

$$F(0) = F(\infty) = 0, \ \Phi(0) = 1{\cdot}0, \ \Phi(\infty) = 0. \quad (22)$$

It should be noted that the condition $F'(0) = 0$ is relaxed to account for the reduction in the order of the momentum equation and that the momentum boundary layer thickness approaches zero as $Pr \to 0$. This limiting solution is in essence an outer solution relative to the thermal field. The surface characteristics given in equations (14) and (15) reduce respectively to

$$\frac{\tau L^2}{\rho v^2} = G\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{3n+1}{4}} \times (Pr)^{\frac{1}{4}}(n + 3)^{-\frac{1}{4}}F''(0) \quad (23)$$

$$\frac{q\beta g L^4}{v^2 k} = -G\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{5n-1}{4}} \times (Pr)^{\frac{1}{4}}(n + 3)^{\frac{1}{4}}\Phi'(0). \quad (24)$$

It is clearly seen that both the wall shear stress and heat transfer, similar to the classical Pohlhausen's solution ($n = 0$), vanish as $Pr^{\frac{1}{2}}$ as the Prandtl number approaches zero.

Introducing

$$\eta = (n + 3)^{-\frac{1}{4}}(Pr)^{-\frac{1}{4}}\xi \tag{25}$$

$$f(\eta) = (n + 3)^{-\frac{3}{4}}(Pr)^{-\frac{3}{4}}F(\xi) \tag{26}$$

$$\theta(\eta) = \Phi(\xi) \tag{27}$$

for $Pr \to \infty$, equations (7) and (8) reduce respectively to

$$F''' + \Phi = 0 \tag{28}$$

$$\Phi'' + F\Phi' - \frac{4n}{n+3}(F')\Phi - \frac{4nm}{n+3}(F') = 0 \tag{29}$$

with the boundary conditions

$$F(0) = F'(0) = F''(\infty) = 0, \; \Phi(0) = 1{\cdot}0, \; \Phi(\infty) = 0. \tag{30}$$

It is also significant to note that $F''(\infty) = 0$ is invoked here instead of $F'(\infty)$ because this limiting solution represents an inner solution relative to the thermal field. The corresponding surface characteristics can be written as

$$\frac{\tau L^2}{\rho v^2} = G\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{3n+1}{4}} \times (n + 3)^{-\frac{1}{4}}(Pr)^{-\frac{1}{4}}F''(0) \tag{31}$$

$$\frac{q\beta g L^4}{v^2 k} = -G\left(\frac{4}{G}\frac{x}{L}\right)^{\frac{5n-1}{4}} \times (n + 3)^{\frac{1}{4}}(Pr)^{\frac{1}{4}}\Phi'(0). \tag{32}$$

It should be noted that the surface heat flux increases without bound as $Pr^{\frac{1}{4}}$ for large and increasing values of the Prandtl number; this is again similar to the case without environmental temperature stratification. In view of the two limiting solutions relative to the parameter Pr, it is highly likely that the characteristic limiting Prandtl-number behavior is inherent in external free convection, and in particular, is not influenced by either the wall temperature or the ambient temperature distributions.

Before detailed results are presented, it is pertinent to point out the unique characteristics of the free-convection problem with environmental temperature stratification; it is possible to have the temperature in the outer region of the boundary layer falling below that of the local environment and the occurrence of flow reversal in this region. Even though this has been observed by Cheesewright [2] and Eichhorn [4], the physical implication of this behaviour has not been clearly discussed. In the present study, a qualitative examination of the governing differential equations shows the physical meaning of the negative θ as well as leads to the specification of the wall and the environmental temperature distributions which are necessary to induce the temperature defect and the flow reversal. When the temperature inside the boundary layer falls below the local ambient temperature T_∞ there must be a region of y or η close to the ambient where $\theta' > 0$. This condition, for y or η sufficiently large but finite, leads to

$$\left[n\theta + \left(\frac{n-1}{4}\right)\eta\theta'\right] < 0$$

provided that $n < 1$, which is true in cases cited by Cheesewright [2] and Eichhorn [4]. This inequality directly implies

$$\left[\theta\frac{\partial(T_w - T_\infty)}{\partial x} + (T_w - T_\infty)\frac{\partial\theta}{\partial x}\right] < 0$$

or simply

$$\frac{\partial T_\infty}{\partial x} > \frac{\partial T}{\partial x}. \tag{33}$$

The physical implication of the inequality (33) is still not clear because T_∞ could either increase or decrease with respect to x in accordance with Fig. 2. It remains to be shown that equation (33) must be interpreted on the basis that $\partial T_\infty/\partial x > 0$. Letting $Pr = 1{\cdot}0$ without any loss of generality, the transformed energy equation reduces to

$$\theta'' + (n + 3)f\theta' = 4n(\theta + m)f'. \quad (34)$$

For sufficiently large but finite η, $f(\eta)$ is nearly a constant and equation (34) has a solution of the form

$$\theta' = \exp[-(n + 3)\eta f]\{\theta'(0) + \int_0^\eta 4n(\theta + m)f' \exp[(n + 3)\eta f]\,d\eta\}. \quad (35)$$

A negative region for θ implies $\theta' = 0$ at a certain $\eta < \infty$. If $\theta'(0)$ is negative,* the integral in equation (35) must be positive at that η. This is only possible when both m and n carry the same sign, for example, either $m > 0$, $n > 0$ or $m < 0$, $n < 0$. In view of Fig. 2, these conditions imply that the negative regions of θ are only compatible with an environment whose temperature increases with x. A similar analysis on the transformed momentum equation shows that these are the same conditions for a flow reversal to occur inside the boundary layer. Since $\partial T_\infty/\partial x$ is positive, the inequality in equation (33) signifies that the temperature rise inside the boundary layer must lag behind that in the environment. In other words, when the ambient temperature rise becomes sufficiently strong, the temperature of the fluid inside the boundary layer tends to lag behind. As a result, a local temperature defect and a flow reversal tend to occur. In addition to the results shown by Cheesewright [2] and Eichhorn [4], this behavior was well substantiated by the extensive numerical calculations carried out for this investigation. Indeed, it has been found that a temperature defect and a flow reversal do not occur in the cases where the environmental temperature decreases with x.

RESULTS AND DISCUSSION

As previously pointed out, the present similarity solution includes three physical parameters; namely, the wall and ambient temperature distribution parameters m and n, and the Prandtl number of the fluid. It has also been noted that n governs the variation of the local temperature difference between the surface and environment. Extensive numerical solutions, based on the technique of Nachtsheim [8], have been carried out for equations (7) and (8). The calculations include values of m and n between $+1$ and -1, and Prandtl numbers between 0·1 and 20. In addition, numerical results have been obtained for the limiting Prandtl number cases for identical values of m and n.

Surface heat-transfer results, as represented by $[-\theta'(0)]$ for Prandtl numbers of 0·72 and 10 are shown in Figs. 3 and 4. It is understood that any physical interpretation of these results must be made in terms of the surface heat flux given by equation (15). To facilitate such interpretation, comparison of the heat transfer results for the various parameters can be made at a given location x/L along the plate with the local temperature difference. On this basis, equation (15) reduces to

$$Nu/Gr_x^{\frac{1}{4}} = \frac{-\theta'(0)}{\sqrt{2}} \quad (36)$$

where Nu is the local Nusselt number based on the characteristic length x and the local temperature difference, and Gr_x is a local Grashof number defined by

$$Gr_x = \frac{g\beta x^3(T_w - T_\infty)}{\nu^2}. \quad (37)$$

Consequently, it is only necessary to compare the $[-\theta'(0)]$ values in discussing the effects of various parameters on the surface heat flux. For surface temperatures higher than that of the environment, $[-\theta'(0)] > 0$ indicates a flow of heat into the fluid whereas negative values of $[-\theta'(0)]$ signify a flow of heat into the surface. For $n = 0$, both the surface and the ambient temperatures remain constant; this is identical to the classical Pohlhausen solution. For $m = 0$, the surface temperature varies and the ambient

* As indicated in Figs. 3 and 4, $\theta'(0)$ can be positive. This is a situation where the wall temperature decreases very rapidly with x and there is a high energy content along $x = 0$. Under these conditions negative θ's and a flow reversal are not evident.

temperature remains constant; this case is identical to that treated by Sparrow and Gregg [5]. The $m = -1$ case dictates a uniform surface temperature with a variable ambient temperature. Also in accordance with equation (15), $n = 0{\cdot}2$ corresponds to a uniform surface heat flux. In addition, for $n = 1$, both temperature distributions are linear.

The physical effect of the parameters m and n on the heat transfer rate is given in Figs. 3 and 4

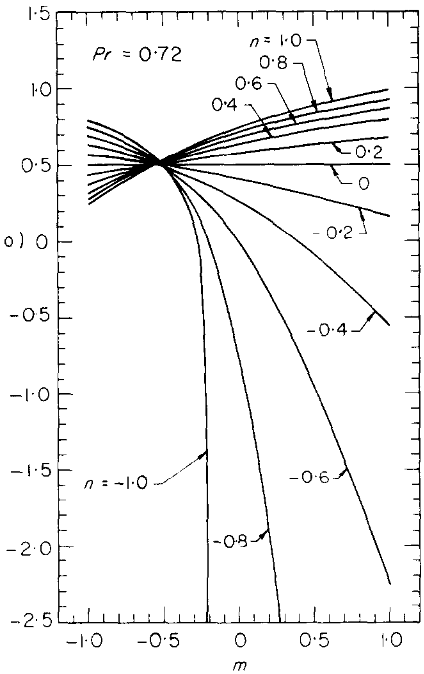

FIG. 3. Dimensionless surface temperature gradient as a function of m and n for $Pr = 0{\cdot}72$.

in terms of $[-\theta'(0)]$. For fixed n, the calculated behavior of $[-\theta'(0)]$ as a function of m is essentially what could be expected physically. For instance, with $m = -1$ and a fixed $n > 0$, the surface temperature remains constant while the ambient temperature decreases with x; the surface heat flux is less than that based on a common local temperature difference and x/L

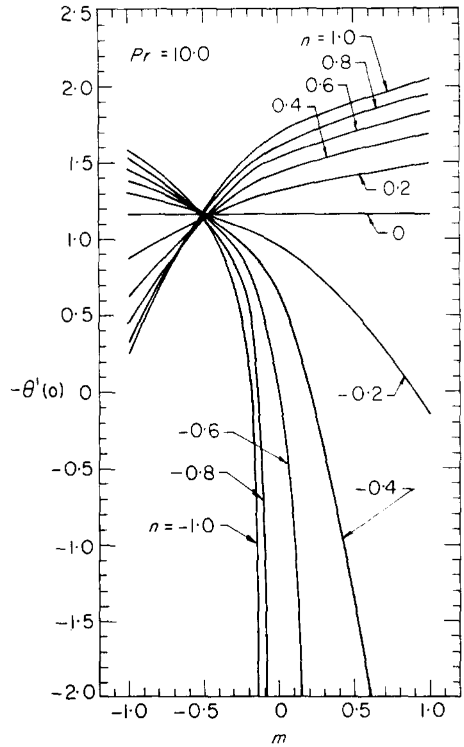

FIG. 4. Dimensionless surface temperature gradient as a function of m and n for $Pr = 10$.

used in the constant wall and ambient temperature solution ($n = 0$). This is primarily caused by the fact that the temperatures in the boundary layer do not completely respond to the decrease in T_∞ resulting in a lower effective temperature difference. As m increases, the surface temperature increases resulting in a higher heat transfer rate since the temperatures in the boundary layer begin lagging the increase in T_w. At $m \approx -\frac{1}{2}$ the two effects virtually balance each other. At $m = 0$, the ambient temperature is uniform and the surface heat transfer is dictated by the surface temperature distribution. For $m > 0$, the increase in surface temperature combines with the increase in the ambient temperature; local effective temperature difference across the boundary layer is greater than the local temperature difference resulting in a still higher heat transfer rate at the surface. This situation is completely reversed for a fixed $n < 0$. At $m = -1$ for a fixed $n < 0$, the surface temperature remains constant while the ambient temperature increases with x; the local temperature difference also decreases with x. Again in view of the fact that the temperatures in the boundary layer lag behind changes in the ambient temperature, the effective temperature difference is higher than the local temperature difference resulting in a higher heat transfer rate than that based on the local temperature difference alone. As m increases, the wall temperature decreases with x resulting in a reduced heat transfer rate. Again, the two effects approximately cancel at $m \approx -$ When m becomes sufficiently high, an additional factor is introduced. The high energy content along $x = 0$, as given by the mathematical description of the similarity solution and caused by factors other than buoyancy forces due to heating, reduces the surface heat flux and eventually overwhelms the normal free convection behavior. This is represented by a transfer of heat to the surface as given by positive values of $\theta'(0)$. This has been observed by Sparrow and Gregg [5] for $m = 0$.

It is also of interest to note in Figs 3 and 4 that there seems to be a symmetry between the cases for $m = -1$ (T_w = constant) and $m = 0$ (T_∞ = constant). At $m = -\frac{1}{2}$, the wall and ambient temperature distributions are mirror images about the reference temperature T_r with T_w decreasing with x and T_∞ increasing with x. As discussed, these distributions generate opposite trends in the surface heat flux; the two effects tend to cancel $m \approx -\frac{1}{2}$. The resulting $[-\theta'(0)]$ values are very close to the results for $n = 0$ (T_w, T_∞ = constants), and this behavior seems to be rather insensitive to Prandtl number. At first, one might believe that the surface heat flux would be passive to changes in the ambient temperature whereas very active with respect to changes in the surface temperature. It is obvious from the previous discussion that this is not entirely the situation. A variation in T_w or T_∞ has about the same effect in causing the temperatures near the surface in the boundary layer to lag behind the changes in T_w or T_∞ either causing an increase or decrease in surface heat transfer. This is true except for the region where $\theta'(0) > 0$.

The effect of n on the surface heat transfer given in terms of equation (15) for a fixed m is somewhat more difficult to interpret physically. However, it can still be properly interpreted by examining the two cases of $m = -1$ (T_w = constant) and $m = 0$ (T_∞ = constant) using the notion of a common local temperature difference as was used previously when comparing the fixed n results to the case of $n = 0$. For $m = -1$, the surface heat transfer, when compared on a common local $(T_w - T_\infty)$ and a common x/L, and decreases with increasing n. This effect reverses itself for $m = 0$; the surface heat flux increases with increasing n. It should be noted that when using this type of comparison, the heat flux, in accordance with equation (36), will follow the value of $[-\theta'(0)]$.

The quantitative results in Figs. 3 and 4 also enable one to discuss the validity of two approximate solutions based on well-known analyses to the problem of variable surface and ambient temperatures. The first, which has been used in the previous paragraphs for comparison purposes, is to treat the present problem by approximating the local heat transfer by that corresponding to the classical constant surface and ambient temperature case (for example, $n = 0$). together with the local values of T_w and T_∞. The error is essentially given by the difference between $q(x;m,n)$ and $q(x;m,0)$. Also, this approximation does not predict the reversal in surface heat transfer. The second approximation is based on the well-known results of Sparrow and Gregg [5]. The local heat transfer in this case is given by the value corresponding to the local temperature difference taken to be a power function of the distance coordinate, $(T_w - T_\infty) \sim x^n$. Here it can be shown that the error is given by the difference between $q(x; m, n)$ and $q(x; 0, n)$. Furthermore, this approximation does not predict the proper behavior of n in the region $m < 0$. The results presented in Figs. 3 and 4

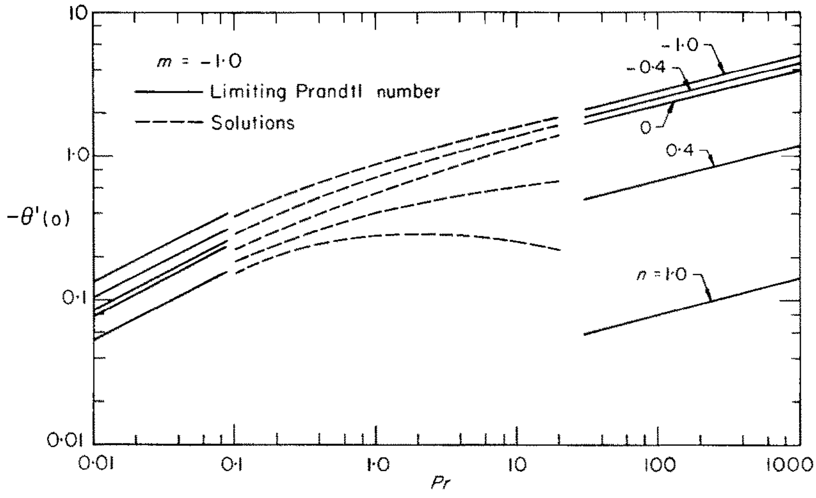

FIG. 5. Dimensionless surface temperature gradient as a function of Pr and n for $m = -1$.

clearly show that the approximate solutions are not adequate to predict the surface heat transfer under the conditions of a varying ambient temperature.

Figures 3 and 4 give the typical behaviour for all Prandtl numbers covered in the numerical calculations, including those of the limiting Prandtl-number solutions. Figures 5 and 6 show the behavior of $[-\theta'(0)]$ as a function of Pr and n over the entire Prandtl-number range for $m = -1$ and $m = 1$. Generally, it is seen that the individually calculated results match with the limiting asymptotes very well. For m and n both positive or negative, $[-\theta'(0)]$ increases monotonically from a behavior proportional to $Pr^{\frac{1}{2}}$ for small Prandtl numbers to

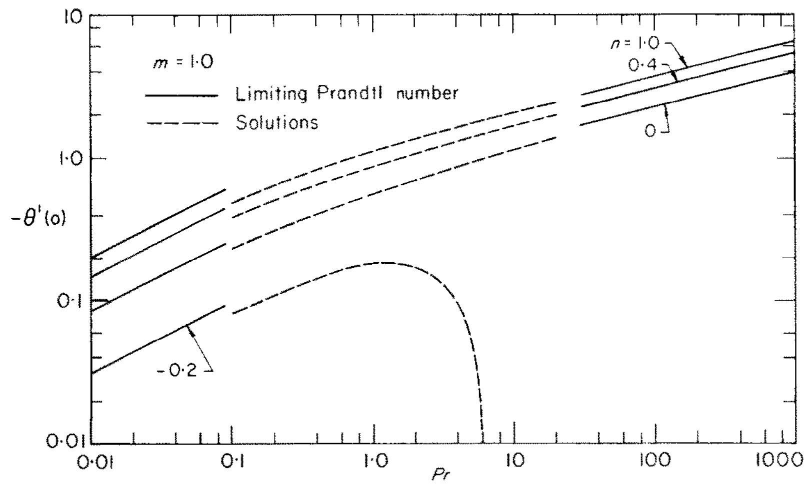

FIG. 6. Dimensionless surface temperature gradient as a function of Pr and n for $m = 1$.

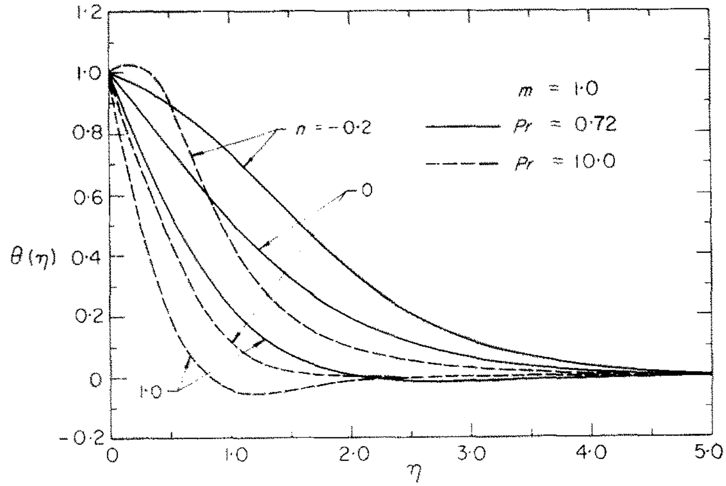

FIG. 7. Dimensionless temperature profiles for $m = 1$.

that of $Pr^{\frac{1}{4}}$ for large Prandtl numbers. It is of interest to note that the limiting curves are even valid at moderate Prandtl number levels. However, for $n < 0$ and sufficiently high values of m, $[-\theta'(0)]$ decreases rapidly; this is shown in Fig. 6 and previously noted in Figs. 3 and 4. Also for $m < 0$ and $n < 0$, the effect of Prandtl-number is rather unique for free-convection situations. As shown in Fig. 5, $[-\theta'(0)]$ first increases at low Pr, reaches a maximum, and then approaches the limiting curve from above. The physical implication of this behavior is not clear at this time.

Several typical sets of temperature profiles are shown in Figs. 7–9. Several of the interesting features discussed previously are apparent in Fig. 7 for several $m = 1$ cases. The temperature defects are quite evident and occur here for $n = 1$. This phenomenon is more profound at higher Prandtl numbers. For $n = -0{\cdot}2$, an inflection appears in the temperature profile signifying a reduced $[-\theta'(0)]$. In fact for $Pr = 10$,

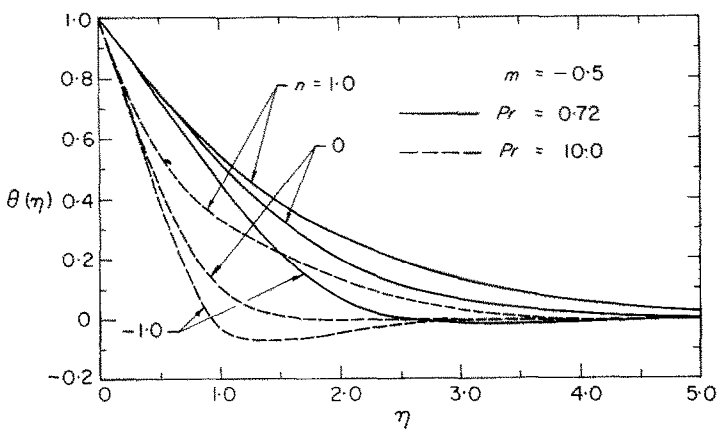

FIG. 8. Dimensionless temperature profiles for $m = -\frac{1}{2}$.

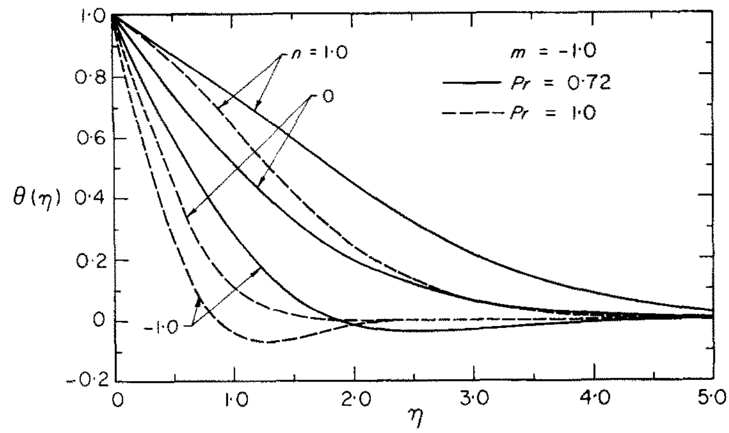

FIG. 9. Dimensionless temperature profiles for $m = -1$.

a heat flux reversal at the surface is evident. Furthermore, the thermal boundary-layer thickness behaves as would be expected for other free-convection situations; it decreases as the Prandtl number increases. Figure 8 shows similar profiles at $m = -\frac{1}{2}$ which are of special interest in view of the crossover behavior for $[-\theta'(0)]$ as indicated in Figs. 3 and 4. A natural question to ask is whether or not the entire thermal field might also be the same for $m = -\frac{1}{2}$ regardless of the value of n. Figure 8 shows that this is indeed not the case; the crossover behavior $[-\theta'(0)]$ is largely a near-surface phenomenon. The temperature defects for $m, n < 0$ are also rather evident; it is more pronounced at the higher Prandtl number. Typical temperature profiles for $m = -1$ which correspond to a constant surface temperature and variable ambient temperature are shown in Fig. 9.

It is significant to mention that in all the calculated cases that contain a temperature defect, flow reversal also takes place in the outer part of the boundary layer. As pointed out in the qualitative analysis, this is expected to occur when $(m, n) > 0$ or $(m, n) < 0$. Physically, this is the direct result of streamwise temperature variation in the boundary layer lagging behind that in the environment when the environment temperature increases in the streamwise direction. The physical implication of this behavior is that here is an additional mechanism to cause hydrodynamic instability inside a boundary layer. Even more interesting, this instability is introduced by a stable temperature stratification in the environment.

CONCLUSIONS

In this paper a similarity solution with sufficient flexibility to describe the surface and ambient temperature variations for laminar free convection along a heated vertical plate is utilized to discuss the relative effects of variable surface and ambient temperatures on the surface heat transfer. Based on extensive numerical calculations, the following conclusions may be drawn:

1. The surface temperature distribution plays about the same role in influencing the surface heat flux as the ambient temperature distribution. The surface heat transfer depends more effectively on a temperature difference which anticipates the variation in the surface or ambient temperatures than the true local temperature difference.

2. For the present variable surface and ambient temperature problem, approximate solutions based on the local temperature difference alone and on the local temperature difference taken as a power function of the distance coordinate are not adequate to quantitatively predict the correct surface heat flux.
3. For the present variable surface and ambient temperature problem, the limiting Prandtl-number effects are the same as that of similar free-convection phenomena. More specifically, the surface heat transfer varies as $Pr^{\frac{1}{2}}$ as $Pr \to 0$ and as $Pr^{\frac{1}{4}}$ as $Pr \to \infty$. One interesting unique Prandtl-number behavior is that for $m < 0$ and $n > 0$ when both temperatures decrease with x, the surface heat flux approaches the large Prandtl-number asymptote from above.
4. When ambient temperature increases with x, i.e. for $(m, n) > 0$ or $(m, n) < 0$, temperature defects and flow reversal inside the boundary layers can be expected; this behavior, induced by a thermally stable environment, provides an additional mechanism to induce an unstable boundary layer.

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CONVECTION NATURELLE LAMINAIRE À PARTIR D'UNE PLAQUE NON-ISOTHERME IMMERGÉE DANS UN MILIEU STATIFIÉ EN TEMPÉRATURE

Résumé—Cette étude concerne le transfert thermique par convection naturelle à partir d'une plaque plane verticale non-isotherme immergée dans un environnement stratifié en température. Des calculs numériques importants basés sur des solutions de similarité one été menés pour un large domaine de distributions de températures pariétales et ambiantes à des nombres de Prandtl entre 0, 1 et 20. On a aussi considéré les cas limites du nombre de Prandtl tendant vers zéro ou vers l'infini. L'effet relatif des distributions de température individuelle a été discuté en détail. On étudie de près le comportement du renversement de l'écoulement et du déficit en température trouvé dans d'autres études relatives à des températures ambiantes non-uniformes. On montre aussi que des approximations basées sur la différence de température locale peuvent introduire de grandes erreurs dans l'estimation des flux thermiques superficiels. Bien que la valeur de cette erreur dépende de l'endroit considéré, le procédé de calcul est clairement discuté.

LAMINARE FREIE KONVEKTION AN EINER NICHTISOTHERMEN PLATTE NACH EINTAUCHEN IN EIN MEDIUM MIT TEMPERATURSCHICHTUNG

Zusammenfassung—Die Arbeit befasst sich mit dem Wärmeübergang durch freie Konvektion an einer nichtisothermen senkrechten ebenen Platte, die in eine Umgebung mit Temperaturschichtung getaucht wurde. Ausgehend von Ähnlichkeitslösungen wurden ausführliche Berechnungen durchgeführt für eine Vielzahl von Wand- und Umgebungstemperaturverteilungen bei Prandtlzahlen zwischen 0,1 und 20. Die Grenzfälle $Pr \to 0$ und $Pr \to \infty$ wurden ebenfalls betrachtet. Der relative Einfluss der einzelnen Temperaturverteilungen wird detailliert erörtert. Die Strömungsumkehr und das Fehlverhalten der Temperatur, wie es bei anderen Untersuchungen gefunden wurde, die sich mit ungleichförmigen Umgebungstemperaturen befassten, wurden genau untersucht. Es wird ebenfalls gezeigt, dass Näherungslösungen, die auf örtlichen Temperaturdifferenzen aufbauen, zu grossen Fehlern bei Voraussagen des Wärmeübergangs an der Oberfläche führen. Obwohl die Grösse des Fehlers von der jeweiligen Situation abhängt, wird das Verfahren zu seiner Berechnung exakt erörtert.

ЛАМИНАРНАЯ СВОБОДНАЯ КОНВЕКЦИЯ НА НЕИЗОТЕРМИЧНОЙ ПЛАСТИНЕ, ПОГРУЖЕННОЙ В СРЕДУ С РАЗНОЙ ТЕМПЕРАТУРОЙ ПО СЛОЯМ

Аннотация—Исследован свободноконвективный перенос тепла от неизотермичной вертикальной плоской пластины, погруженной в среду с разной температурой по слоям. Выполнено большое количество численных расчётов автомодельных задач для разных распределений температуры стенки и окружающей среды в диапазоне чисел Прандтля от 0,1 до 20. Рассмотрен предельный случай, когда число Прандтля стремится к нулю и бесконечности, а также относительный эффект некоторых распределений температуры. Подробно исследованы характеристики обратного течения и отклонений температуры, полученные в других работах по исследованию неоднородной температуры окружающей среды. Показано также, уто использование аппроксимаций, основанных на локальной разности температур может приводитв к большим погрешностям при расчёте скорости переноса тепла на поверхности. Представлена методика расчёта данной погрешности, несмотря на то, что её величина зависит от конкретного исследуемого случая.